ON PARTIALLY ORDERED RINGS

E.E. Shirshova

Moscow, Russia

Let G be a partially ordered additive group [1]. An element $x \in G$ is said to be *very much less than* $y \in G$ $(x \ll y)$ if $nx \leqslant y$ for all $n \in \mathbb{Z}$. We denote by M_a the set of all $g \in G$ such that $g \ll a$ for some $a \in G$, where $a > 0$.

Definition 1. *A ring R is called a partially right $\mathcal{K}$-ordered ring, if it satisfies the conditions: 1) $< R, + >$ is a partially ordered group; 2) if $a > 0$ in R, then $ar \ll a$ for all $r \in R$. If the order of the group $< R, + >$ is linear, we say that the ring R is a linear right $\mathcal{K}$-ordered ring.*

Theorem 2. *Let R be a ring, I being an ideal of R satisfying the conditions: 1) the rings $< I, \leqslant_1 >$ and $< R/I, \leqslant_2 >$ are partially right $\mathcal{K}$-ordered rings; 2) the conditions $a \in I$ and $a >_1 0$ imply the inequality $ar \leqslant_1 a$ for each $r \in R$. Then the ring R can be furnished with a partial right $\mathcal{K}$-order $\leqslant$, from which the order $\leqslant_1$ inherits in the ring I, and the order $\leqslant_2$ inherits in the ring R/I; and I being a convex ideal of R under the order $\leqslant$.*

Let $\mathfrak{U}$ be a system of ideals of a ring R, including $\{0\}$ and R. The system of ideals $\mathfrak{U}$ is called *complete*, if for any subsystem of $\mathfrak{U}$ the following is true: both union and intersection of ideals forming this subsystem belong to $\mathfrak{U}$. An inclusion of ideals $A \subset B$ are said *to be a step*, if the relation $A \subseteq J \subseteq B$ implies $A = J$ or $J = B$ for each ideal J of R.

Theorem 3 *If R is a linear right $\mathcal{K}$-ordered ring, then there exists a complete system of right ideals in R, where any step of right ideals $I \subset J$ in this system satisfies the conditions: 1) $JR \subset I$; 2) I is a convex ideal of J; 3) $I = M_b$ for all $b \in J \setminus I$, where $b > 0$.*

Theorem 4 *Let R be a ring without an identity element, and there exists a complete system of right ideals in R which satisfies the condition: if $I \subset J$ is any step of right ideals in this system, then I is an ideal of J, and the quotient group $< J/I, + >$ is a torsion-free group. Then R can be made into a linear right $\mathcal{K}$-ordered ring.*

References

1. Fuchs L. *Partially ordered algebraic systems*. Oxford - London - New York - Paris: Pergamon Press, 1963.